Sec 6.2, 6.6 Two Dimensional Constant Linear Systems (Phase Plane)

In this section we will only consider linear systems of the form $\vec{Y}' = A\vec{Y}$, where A is an invertible 2×2. These systems can be written as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$\vec{y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$

$\vec{y}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$

where $\det(A) \neq 0$.

How to sketch the solutions of this system?

Ex1. Consider the following first order linear system: $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

The general solution is given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \mathbf{c}_1\mathbf{e}^{-1t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mathbf{c}_2\mathbf{e}^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Notice that the eigenvalues are real and negative.

Step1: Sketch the four "easy" solutions; i.e. the solutions that correspond to the pairs:

$\mathbf{c}_1$	$\mathbf{c}_2$
1	0
-1	0
0	1
0	-1

For example, if $\mathbf{c}_1 = 1$ and $\mathbf{c}_2 = 0$, then we can identify the corresponding solution with the pair

$$(x(t), y(t)) = \mathbf{e}^{-t}(1, 0).$$

This is the parametric curve $x(t) = \mathbf{e}^{-t}$, $y(t) = 0$. The **trace** of this parametric curve is the positive x-axis.

For example, if $\mathbf{c}_1 = 0$ and $\mathbf{c}_2 = 1$, then we can identify the corresponding solution with the pair

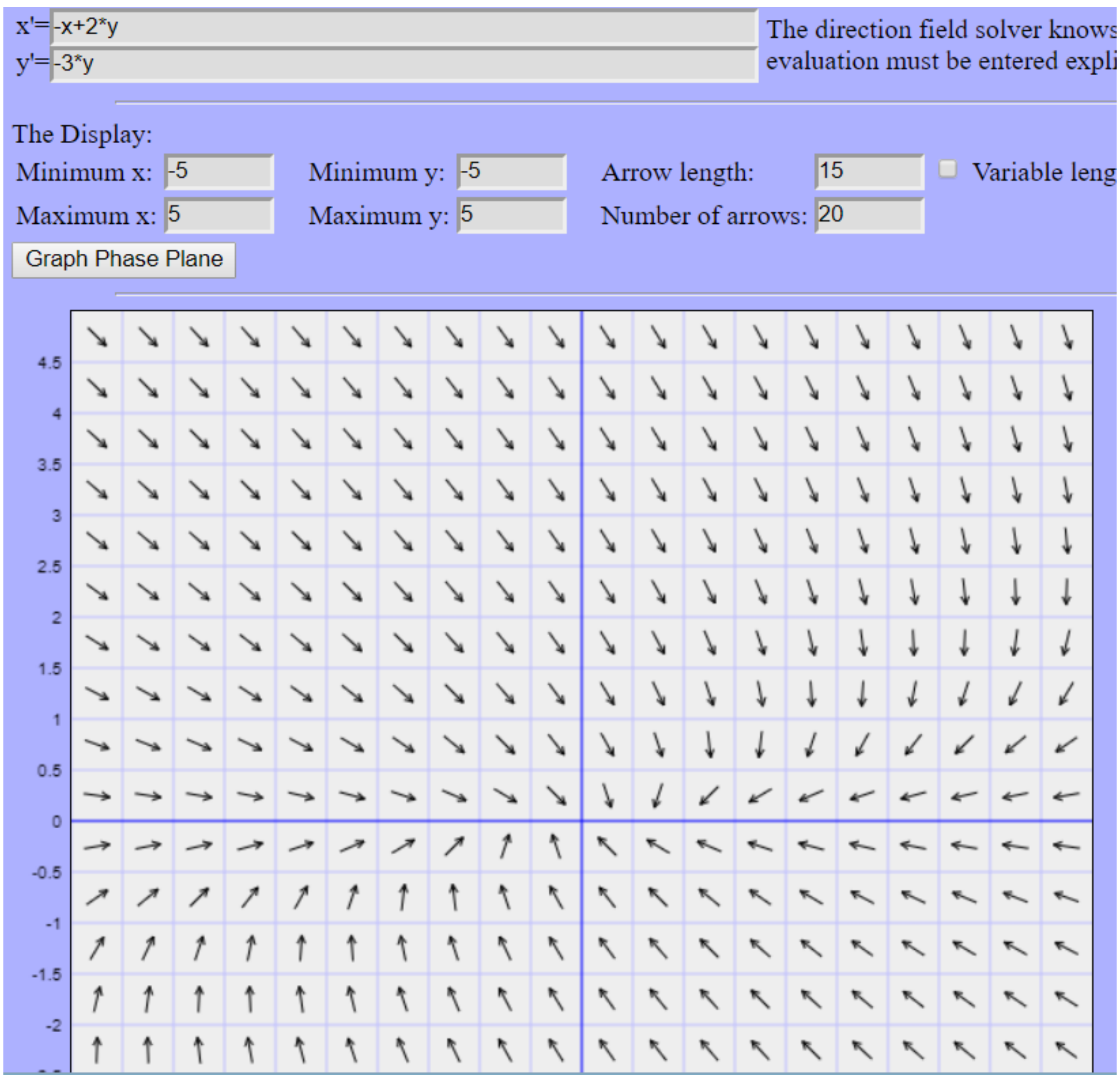
$$(x(t), y(t)) = \mathbf{e}^{-3t}(1, -1).$$

This is the parametric curve $x(t) = \mathbf{e}^{-3t}$, $y(t) = -\mathbf{e}^{-3t}$. The **trace** of this parametric curve is the open ray $y = -x$ whose end point at the origin and contains the point $(1, -1)$.

Step2: Fill in the rest. **How?**

We need to understand the behavior of the solutions. Since **both eigenvalues are negative**, regardless the values of $\mathbf{c}_1$ and $\mathbf{c}_2$ any solution converges to the origin point $(0, 0)$. But we can say more about the behavior of the solutions.

- As $t \to \infty$, the dominant term is $c_1 \mathbf{e}^{-1t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. In other words, for large t, any solution is approximately parallel to $\mathbf{e}^{-1t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.
- As $t \to -\infty$, the dominant term is $c_2 \mathbf{e}^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. In other words, for large t (negative) , any solution is approximately parallel to $\mathbf{e}^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.



If the eigenvalues are real and negative we say that the origin is a nodal sink (sink node), or asymptotically stable node.

Ex. 1 (page above) Find Eigen pairs

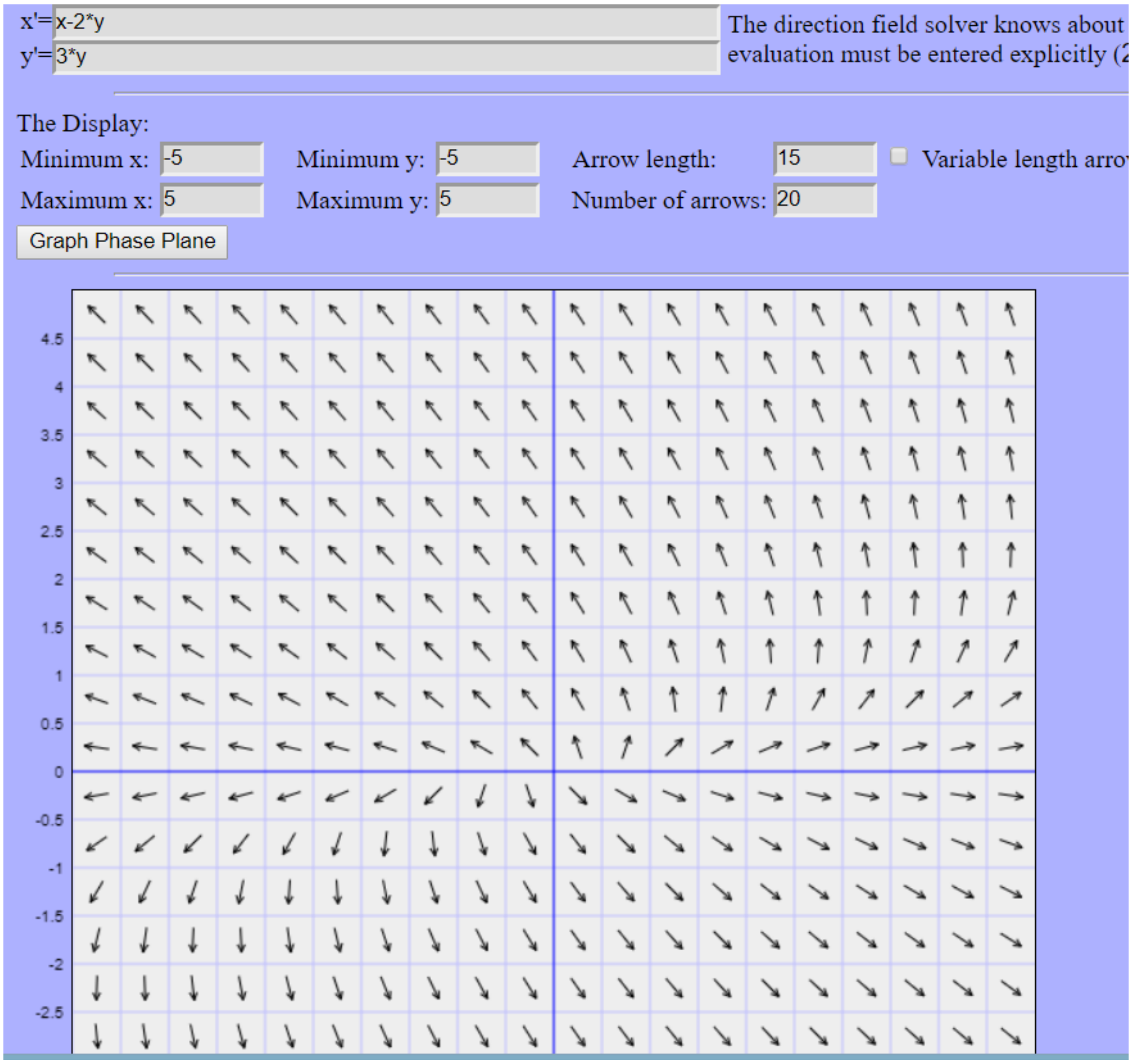
$\vec{y}' = A\vec{y}$ $A = \begin{bmatrix} -1 & 2 \\ 0 & -3 \end{bmatrix}$

① Eigen values $P(\lambda) = \det\begin{pmatrix} -1-\lambda & 2 \\ 0 & -3-\lambda \end{pmatrix} = (1+\lambda)(3+\lambda) = 0$ $\nearrow \lambda_1 = -1$, $\searrow \lambda_2 = -3$

Ex2. Sketch the solutions of the system $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.

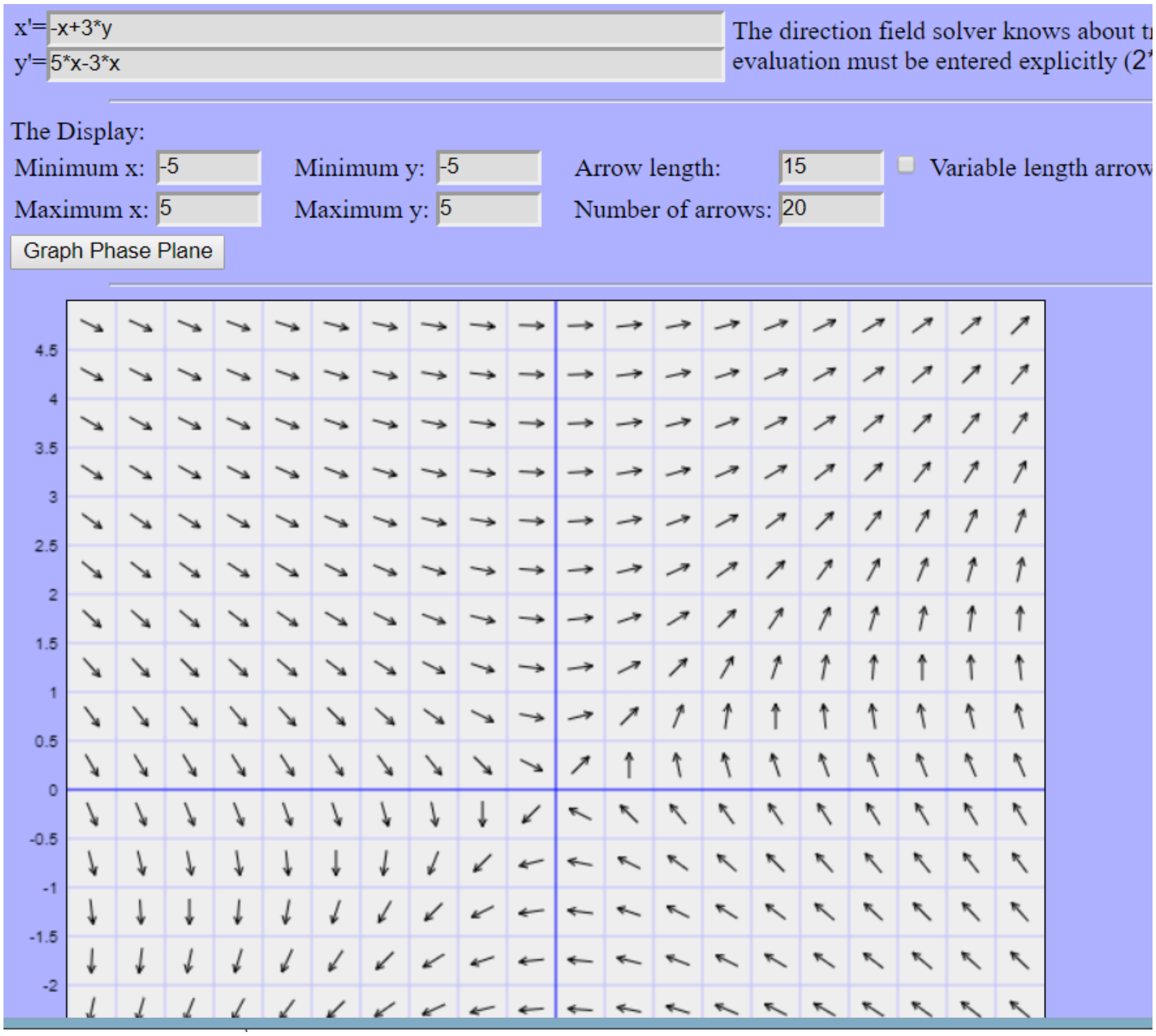
Sol: One sees that the general solution is given by

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \mathbf{c}_1 \mathbf{e}^{1t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \mathbf{c}_2 \mathbf{e}^{3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$



If the eigenvalues are real and positive we say that the origin is a nodal source (source node), or unstable node.

Ex.3 Sketch the solutions of the system $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 5 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$



If the eigenvalues are real and have opposite signs we say that the origin is a saddle point.

Ex 3. Eigenvalues $p(\lambda) = \det\begin{pmatrix} -1-\lambda & 3 \\ 5 & -3-\lambda \end{pmatrix} = 0$

$$p(\lambda) = \underbrace{(1+\lambda)(3+\lambda)}_{\lambda^2+4\lambda+3} - 15 = 0 \Leftrightarrow p(\lambda) = (\lambda+6)(\lambda-2)$$

$\lambda^2 + 4\lambda - 12$

$\lambda_1 = -6 \Rightarrow \vec{v}_1$

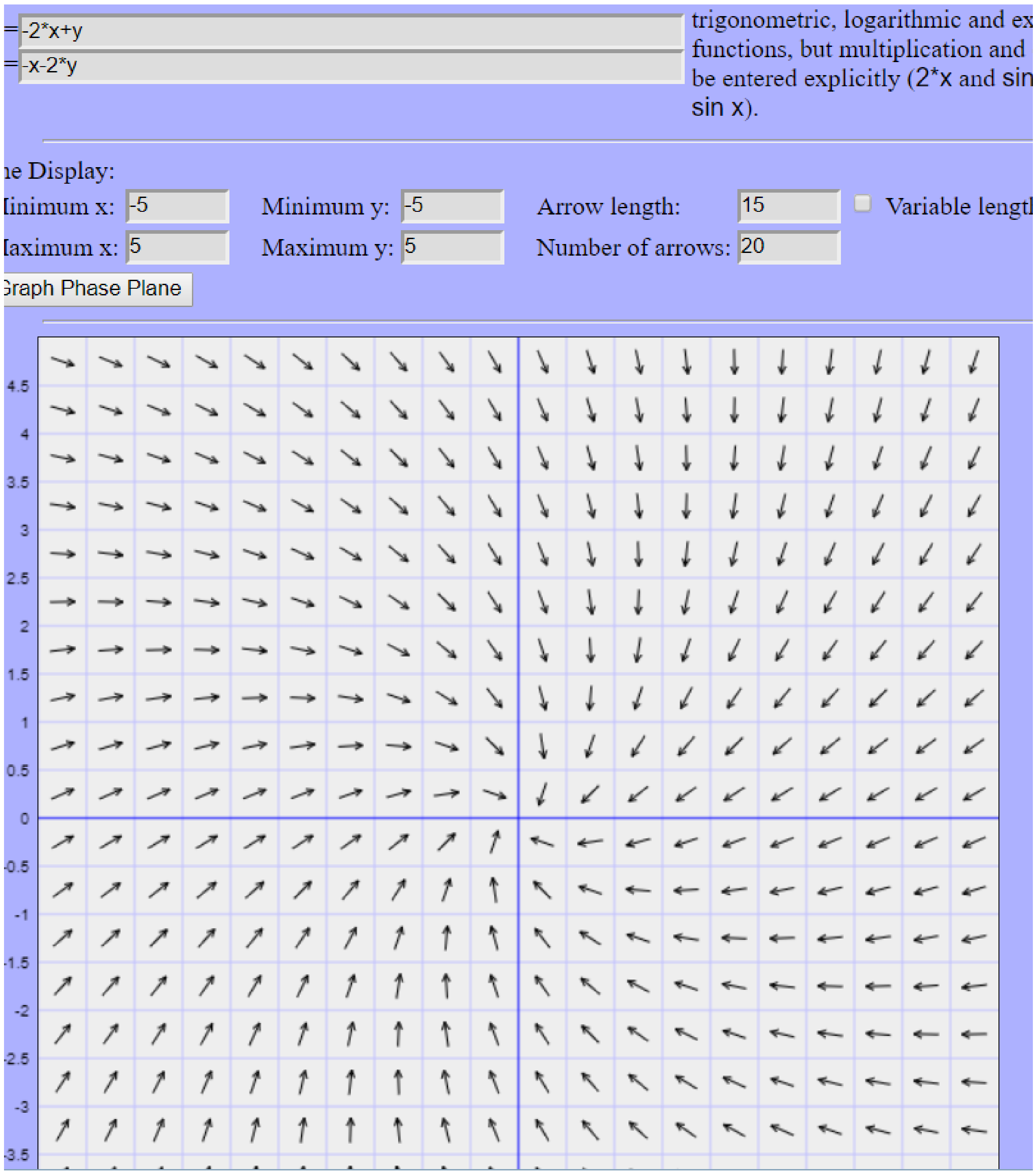
$\lambda_2 = 2 \Rightarrow \vec{v}_2$

Eigenvectors

$$\vec{y}(t) = c_1 e^{-6t}\vec{v}_1 + c_2 e^{2t}\vec{v}_2$$

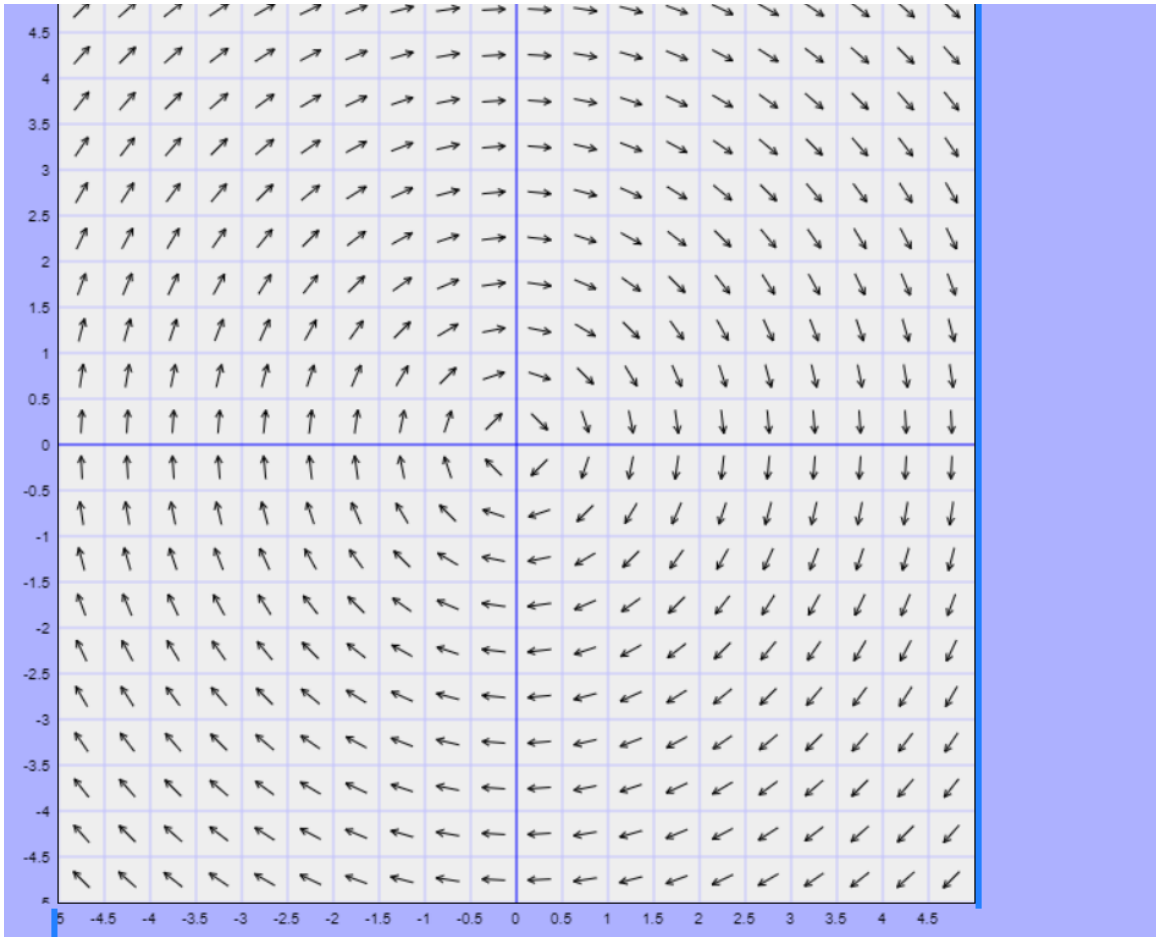
$\vec{v}_1 = \begin{pmatrix} 1 \\ -5/3 \end{pmatrix}$ $\vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Ex.4 Sketch the solutions of the system $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.



Sol. The eigenvalues are $-2 + 1i$ and $-2 - 1i$.

Ex.5 Sketch the solutions of the system $\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$.



Sol. The eigenvalues are $3i$ and $-3i$.

Summary. Let A be an invertible 2×2 matrix. Classification of the origin for $\vec{Y}' = A\vec{Y}$.

Test 2

- Real eigenvalues, both negative: **asymptotically stable node, or sink node**.
- Real eigenvalues, both positive: **unstable node, or source node**.
- Real eigenvalues of opposite sign: **saddle point**.

Test 3

- Complex eigenvalues, $a \pm bi$ with $a < 0$: **asymptotically stable focus**.
- Complex eigenvalues, $a \pm bi$ with $a > 0$: **unstable focus**.
- Complex eigenvalues, $a \pm bi$ with $a = 0$: **center**.